

Controllable Complex Freezing Dynamics Simulation on Thin Films

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Fig. 1. We introduce a novel method for simulating controllable complex freezing dynamics on thin films. We demonstrate the effectiveness of our method through various examples, including the symmetry breaking effects, a swaying dome-like bubble, spiral dendritic crystals guided by fixed orientation, and a freezing thin film inside the 5-point star-shaped boundary (from left to right).

The freezing of thin films is a mesmerizing natural phenomenon, inspiring photographers to capture its beauty through their lenses and digital artists to recreate its allure using effects tools. In this paper, we present a novel method for physically simulating the intricate freezing dynamics on thin films. By accounting for the influence of phase and temperature changes on surface tension, our method reproduces Marangoni freezing and the “Snow-Globe Effect”, characterized by swirling ice dendrites on the film. We introduce a novel Phase Map method on top of the state-of-the-art Moving Eulerian-Lagrangian Particles (MELP) meshless framework, enabling dendritic crystal simulation on mobile particles and offering precise control over freezing patterns. We demonstrate that our method is able to capture a wide range of dynamic freezing processes of soap bubbles and is stable for complex boundaries in our experiments.

CCS Concepts: • **Computing methodologies** → **Physical simulation**.

Additional Key Words and Phrases: freezing dynamics, Eulerian-Lagrangian methods, dendritic crystal simulation, Marangoni flow, artistic control

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1 INTRODUCTION

The realm of phase transition encompasses a myriad of real-world phenomena, including freezing, solid melting, and gasification, among others. Amidst these captivating physical scenarios, the process of bubble freezing emerges as one of the most visually stunning phenomena. While droplets or puddles typically freeze from the propagation of a single freeze front, the freezing of soap bubbles introduces an enchanting spectacle where a profusion of growing ice crystals swirls around, evoking an effect visually reminiscent of a snow globe. Ahmadi et al. [2019] have characterized the underlying physics governing the freezing of bubbles, unveiling two distinct conditions of freezing, i.e., isothermal conditions resulting in the aforementioned snow globe effect, and room-temperature conditions leading to a bottom-up freeze front.

The simulation of the bubble and film freezing process presents a multifaceted challenge that can be delineated into three distinct aspects: 1) the freezing dynamics involving phase transition and thermal conduction, 2) the coupling of the two phases (ice and fluid), and 3) the control of freezing patterns. While no prior work has specifically addressed the simulation of thin film freezing dynamics, researchers in the fields of computer graphics and computational physics have devoted attention to the simulation of dendritic crystal formation and thin film behavior. In the context of thin film simulation, Ishida et al. [2020, 2017] developed dynamic mesh methods to capture the topological evolution of films and foam dynamics. Particle-based methods have been developed, demonstrating proficiency in simulating the turbulent interfacial flow and effectively handling bubble deformation and burst [Deng et al. 2022; Wang

et al. 2020, 2021]. In the domain of dendritic crystal and ice formation simulation, the phase-field theory for crystal growth has been developed [Kharitonsky and Gonczarowski 1993; Kim et al. 2004; Kim and Lin 2003]. Building upon this foundation, Kim et al. [2006] modeled the water freezing process, and Ren et al. [2018] introduced the orientation field to extend the model for arbitrary pattern control of dendritic crystallization. These well-established techniques provide a robust foundation for addressing the simulation of soap bubbles and film freezing within a physically-based framework, but their crystallizations are static in space and do not show complex dynamic effects such as the Marangoni freezing and vigorous swirling ice dendrites on soap films.

In the graphics community, a state-of-the-art meshless bubble motion simulation framework that captures dynamic in-surface flows is the Moving Eulerian-Lagrangian Particles (MELP) method [Deng et al. 2022]. It uses two layers of Eulerian and Lagrangian particles for the motions of bubble shape and in-surface flow. However, it does not have the ability to handle freezing effects due to the absence of the temperature equation, and to difficulties in accurately calculating structured phase transition over their particle representation. In this work, we first re-derive the governing equations of MELP allowing for variable temperature T , then propose a novel Phase Map scheme that is able to generate structured freezing patterns over unstructured (and often locally sparse) particle distributions of the MELP method, enabling highly realistic freezing dendrite formation on the bubble. Moreover, the previous MELP faces stability issues at the boundaries, possibly produced by newly formed dendrites, around the Lagrangian particles. We derive a moving least squares (MLS) concentration boundary formulation for the Eulerian-Lagrangian particle representation of MELP, greatly enhancing the stability during freezing simulations.

In summary, our main contribution includes:

- A novel theoretic framework capable of capturing the thermodynamic mechanisms inherent in the soap bubble and film freezing process.
- A novel Phase Map scheme capable of reproducing structured freezing patterns over moving particle representation, which also allows direct artistic control over visual simulations.
- Derivation and integration of the MLS concentration boundary for the Eulerian-Lagrangian particle representation of MELP, greatly improving the stability near complex boundaries.

2 RELATED WORK

Thin-film modeling. The simulation of thin films has garnered significant attention in various research endeavors, driven by its inherent allure. Depending on the underlying approach employed, including spherical-coordinate grids [Hill and Henderson 2016; Huang et al. 2020], meshes [Da et al. 2015, 2016; Ishida et al. 2020, 2017; Saye and Sethian 2013, 2016; Wang and Chern 2021; Zhu et al. 2014], particles [Deng et al. 2022; Wang et al. 2021], and hybrid structures [Chen et al. 2021; Hyde et al. 2020; Tao et al. 2024]. Particularly, the data structure aspect of our approach draws inspiration from the work by Deng et al. [2022]. By utilizing two kinds of particles tracking

Table 1. Notations used in this paper.

Notation	Meaning
$\mathcal{L}$	the set of Lagrangian particles
$\mathcal{E}$	the set of Eulerian particles
$\mathcal{M}$	the set of Phase Map particles
$\mathcal{C}$	the set of crystals consisting of Lagrangian particles
η	thickness
ζ	phase value
Γ	surfactant concentration
$\mathbf{n}$	normal vector
T	temperature
∇_s	surface gradient operator
θ_{ori}	local molecular orientation
$\mathbf{x}_A$	position of particle $A \in \mathcal{L} \cup \mathcal{E} \cup \mathcal{M}$
a_A	controlling area of A
$\mathcal{N}^B(A)$	all B neighbors in the neighborhood of A
$\mathcal{T}_{A \leftarrow B}(q)$	collect q from $\mathcal{N}^B(A)$ in a spatially averaged way

thin film deformation and surface flow respectively, it achieves state-of-the-art details in meshless methods, a crucial factor for capturing the intricate swirling motion exhibited by ice flakes. We recommend that readers acquaint themselves with this paper beforehand. The section on thin films in the survey by Wang et al. [2024] is also helpful for understanding the research trends of thin film simulation in recent years.

Dendritic pattern control. Dendritic crystallization has been studied by graphics research from both artistic and physical perspectives. Lipton et al. [2013] built a Houdini-based system that allowed artists to have precise control over the frost’s animation and behavior. Utilizing phase-field theory [Barles et al. 1993], Kobayashi [1993] proposed a phase-field model for dendritic crystal growth and successfully generated dendritic patterns. Based on this, Kim and Lin [2003] designed a user-controllable method, Kim et al. [2004] combined phase-field model with diffusion-limited aggregation (DLA) model and improved user control, and Ren et al. [2018] designed a novel orientation field model for arbitrary control of dendritic patterns. Besides the phase-field theory, other theories have been used to simulate the crystallization process. For example, Canabal et al. [2020] proposed a reaction-diffusion model to generate more organic patterns, and Xue et al. [2020] introduced a non-Fourier diffusion solver to simulate the formation of dendrites and snowflakes. In our method, we employ the orientation-based phase-field model to incorporate the MELP method to simulate the thin film freezing process.

Fluid-rigid coupling. Addressing fluid-rigid coupling poses a considerable challenge across various tasks. For Lagrangian coupling methods, Becker et al. [2009] enhanced rigid-fluid coupling results through a predictor-corrector scheme for both velocity and position. Akinci et al. [2012] employed a single layer of boundary samples, applying it to the coupling of fluids with rigid bodies. Density maps [Koschier and Bender 2017] and Moving Least Squares (MLS) [Band et al. 2018] have been introduced for robust boundary

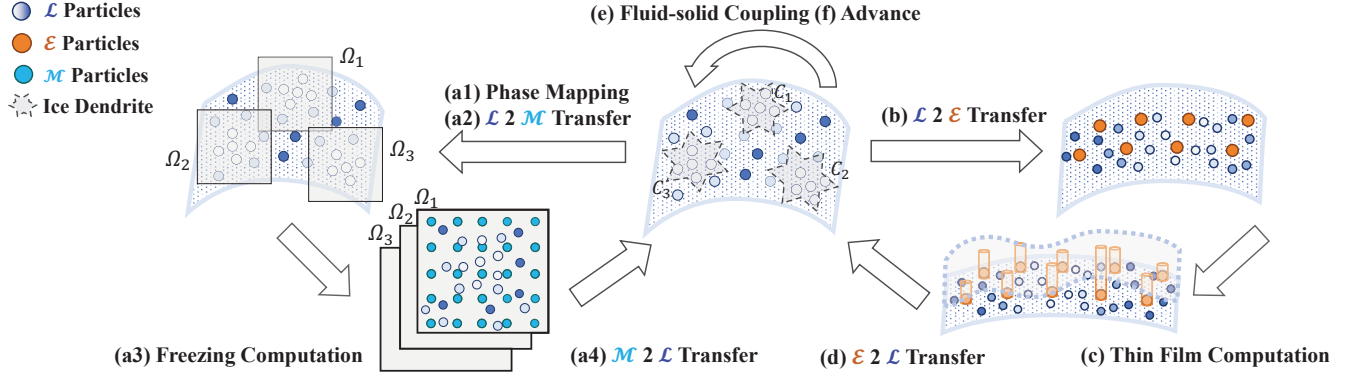


Fig. 2. The pipeline of our Phase Map scheme. (a1)~(a4) Map $\mathcal{L}$ particles onto the context of $\mathcal{M}$ particles and then evolve the freezing substeps. (b)~(d) Transfer quantities from $\mathcal{L}$ to $\mathcal{E}$, perform the thin film computation based on our derivation of the governing equations and the MLS concentration boundary formulation, and transfer quantities from $\mathcal{E}$ back to $\mathcal{L}$. (e) Handle the fluid-solid coupling on $\mathcal{L}$. (f) Advect $\mathcal{E}$ and $\mathcal{L}$ particles.

handling within the Smoothed Particle Hydrodynamics (SPH) framework. Gissler et al. [2019] proposed a unified global formulation that integrates rigid body and fluid dynamics. In our approach, we adopt the MLS pressure boundaries and SPH method for fluid-rigid coupling.

3 FREEZING THIN-FILM SIMULATION

In this section, we formulate our freezing thin-film simulation method. The main notations are listed in Table 1. As a start point, we first employ a set of sparse Eulerian particles $\mathcal{E}$ and a set of dense Lagrangian particles $\mathcal{L}$, which collaborate in both representing the geometry and solving the dynamic equations as is done in MELP [Deng et al. 2022].

3.1 Geometry and Dynamics

We adopt the definition of the geometry of a thin film lamella in the MELP method and expand it into a 6-tuple: $(S_B, \mathbf{R}, H, g, \eta, \zeta)$ to characterize a freezing thin film lamella L . We first designate one of the two air-liquid interfaces as the base surface S_B . Then, for each point $p \in S_B$, we define the field of local frames $\mathbf{R} : S_B \rightarrow SO(3)$, as $\mathbf{R}(p) = \{\mathbf{e}_1(p), \mathbf{e}_2(p), \mathbf{n}(p)\}$ with coordinates $\mathbf{x}^R = (u, v, z)^T$. Additionally, we define fields for mean curvatures $H : S_B \rightarrow \mathbb{R}$, metric tensors $g : S_B \rightarrow \mathbb{R}^{2 \times 2}$, thickness $\eta : S_B \rightarrow \mathbb{R}$, and phase value $\zeta : S_B \rightarrow \mathbb{R}$. The other interface S_T can be defined as the image $f(S_B)$ of the function $f : S_B \rightarrow \mathbb{R}^3$ with $f(p) = p + \eta(p)\mathbf{n}(p)$. Due to the film's minute thickness scale, we assume the quantity at $f(p) \in S_T$ equals that at $p \in S_B$ and specify the geometric quantities on S_B .

Our derivation starts with the thin film dynamics model mentioned by Deng et al. [2022]:

$$\begin{cases} \rho \frac{D\mathbf{u}^\perp}{Dt} = -\frac{\partial p}{\partial z} \mathbf{n} + \delta_I \sigma H \mathbf{n} + \mathbf{f}_{\text{ext}}^\perp, \\ \rho \frac{D\mathbf{u}^\top}{Dt} = \delta_I \nabla_s \sigma + \mathbf{f}_{\text{ext}}^\top, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \quad (1)$$

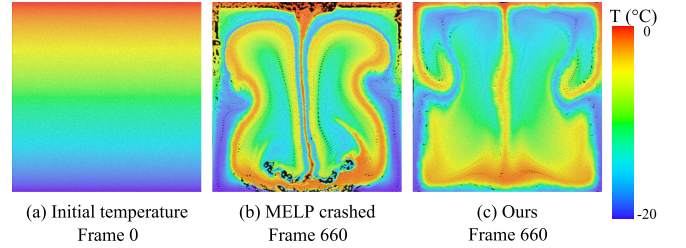


Fig. 3. Temperature-driven Marangoni flow with color indicating the temperature, (a) initialized with a temperature gradient from -20°C (bottom) to 0°C (top). (b) MELP crashed due to the large temperature gradient. (c) Ours can handle it with the comprehensive model considering temperature.

where the superscripts $\perp$ and $\top$ denote normal components and tangential components, ρ the density, $\mathbf{u}$ the velocity, p the pressure, $\mathbf{f}_{\text{ext}}$ the external force, H and $\mathbf{n}$ the mean curvature and the normal vector on the interface, ∇_s the surface gradient operator, $\nabla \cdot \mathbf{u}$ the volumetric divergence of $\mathbf{u}$, and δ_I the Dirac delta function that is non-zero only on the interface. The surface tension coefficient σ is related to the surfactant concentration Γ and temperature T as $\sigma = \sigma_0 - \bar{R}T\Gamma$, where σ_0 is the surface tension for pure water and $\bar{R}$ the ideal gas constant [Xu et al. 2006].

In contrast to previous works where temperature was either overlooked [Ishida et al. 2020, 2017] or treated as constant [Deng et al. 2022; Wang et al. 2021], temperature change plays a pivotal role in driving the swirling motion of ice dendrites. This is attributed to the latent heat released during solidification, which induces temperature variations and subsequently triggers surface tension gradients, known as Marangoni flow [Ahmadi et al. 2019]. We start with the consideration of the surface tension gradient $\nabla_s \sigma$, which reflects the Marangoni effect:

$$\nabla_s \sigma = -\bar{R}(T \nabla_s \Gamma + \Gamma \nabla_s T). \quad (2)$$

Subsequently, we incorporate conservation equations for thickness η and surfactant concentration Γ , along with evolution equations for temperature T and the phase field ζ , to expand Eq. (1). This leads to the following form of the comprehensive freezing dynamics model:

$$\begin{cases} \frac{Du^\perp}{Dt} = -\frac{1}{\rho} \frac{\partial p}{\partial z} \mathbf{n} + \frac{\delta_I(\sigma_0 - \bar{R}T)H}{\rho} \mathbf{n} + \frac{1}{\rho} \mathbf{f}_{\text{ext}}^\perp, \\ \frac{Du^\top}{Dt} = -\frac{2\bar{R}}{\rho\eta} (T\nabla_s \Gamma + \Gamma\nabla_s T) + \frac{1}{\rho} \mathbf{f}_{\text{ext}}^\top, \\ \frac{D\Gamma}{Dt} = -\Gamma\nabla_s \cdot \mathbf{u}, \\ \frac{D\eta}{Dt} = -\eta\nabla_s \cdot \mathbf{u}, \\ \frac{DT}{Dt} = \frac{\partial T}{\partial t} - T\nabla_s \cdot \mathbf{u}, \\ \frac{D\zeta}{Dt} = \frac{\partial \zeta}{\partial t} - \zeta\nabla_s \cdot \mathbf{u}. \end{cases} \quad (3)$$

To solve the above model, we first derive an implicit equation of Γ by reorganizing the tangential part of Eq. (3):

$$\begin{aligned} & \left(\Delta t \frac{2\bar{R}}{\rho} \left(\nabla_s \frac{1}{\eta^*} \cdot \nabla_s T^* + \frac{1}{\eta^*} \nabla_s^2 T^* \right) - \frac{1}{\Delta t \Gamma^*} \right) \Gamma \\ & + \Delta t \frac{2\bar{R}}{\rho} \left(\nabla_s \frac{1}{\eta^*} T^* + \frac{1}{\eta^*} 2\nabla_s T^* \right) \cdot \nabla_s \Gamma + \Delta t \frac{2\bar{R}}{\rho} \frac{1}{\eta^*} T^* \nabla_s^2 \Gamma \\ & = \nabla_s \cdot \mathbf{u}^\top - \frac{1}{\Delta t} + \Delta t \left(\nabla_s \frac{1}{\rho} \cdot \mathbf{f}_{\text{ext}}^\top + \frac{1}{\rho} \nabla_s \cdot \mathbf{f}_{\text{ext}}^\top \right), \end{aligned} \quad (4)$$

where $\mathbf{u}^\top$, Γ^* , η^* and T^* are respective quantities after advection, collected during the $\mathcal{L}\mathcal{E}$ step (mentioned in Section 6). For more details on derivation, we refer readers to the supplement. By solving Eq. (4) with IISPH, we can update the tangential component of $\mathbf{u}$ as:

$$\frac{Du^\top}{Dt} = -\frac{2\bar{R}}{\rho\eta} (T\nabla_s \Gamma + \Gamma\nabla_s T) + \frac{1}{\rho} \mathbf{f}_{\text{ext}}^\top. \quad (5)$$

It should be noted that temperature is calculated in Kelvin units, though displayed in Celsius degrees in the context of this paper.

Compared to the MELP method, our comprehensive freezing dynamics model can cope with large temperature gradients on the thin film as depicted in Fig. 3.

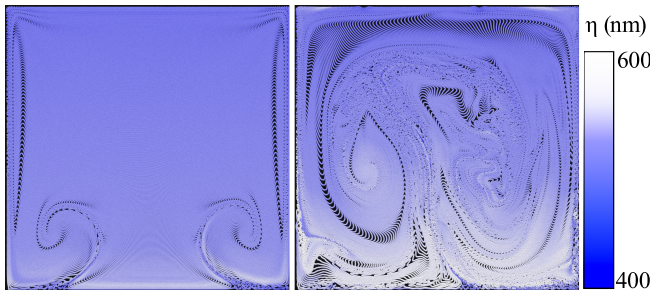


Fig. 4. Marangoni effect induced by temperature changes resulting from latent heat release during solidification, initialized with uniform temperature and colored by thickness. **Left:** Slight surface flow without latent heat. **Right:** Significant surface flow corresponding to the Marangoni effect induced by latent heat. The phase transition is omitted for better observation.

3.2 Orientation-based Dendritic Crystal Simulation

We adopt the method outlined by Ren et al. [2018] to model dendritic crystals based on the phase-field theory. The phase-field theory represents the material's phase by the phase value ζ ranging from 0 to 1. A value of 0 represents liquid, and 1 represents solid. Each phase is driven by free energy to spontaneously evolve towards two stable states, i.e., $\zeta = 0$ and $\zeta = 1$. We assume that the phase transition from liquid to solid on the freezing thin film occurs at the surface. By employing the local frame $R(\mathbf{e}_1, \mathbf{e}_2, \mathbf{n})$ with coordinates $\mathbf{x}^R = (u, v, z)^\top$ and applying surface gradient operator ∇_s , we can derive the equation for phase-field evolution on thin film following Ren et al. [2018] as:

$$\frac{\partial \zeta}{\partial t} = M_\zeta \left(\nabla_s \cdot (\varepsilon^2 \nabla_s \zeta) - \frac{\partial}{\partial u} \left(\varepsilon \frac{\partial \varepsilon}{\partial \theta} \frac{\partial \zeta}{\partial v} \right) + \frac{\partial}{\partial v} \left(\varepsilon \frac{\partial \varepsilon}{\partial \theta} \frac{\partial \zeta}{\partial u} \right) - g'(\zeta) - p'(\zeta)(f_s - f_l + f_{ori}) \right), \quad (6)$$

where M_ζ is a constant affecting solidification rate, ε is the anisotropy function, θ represents the local phase front direction, g and p are functions related to phase value ζ , f_s and f_l are functions related to temperature T , and f_{ori} related to the local molecular orientation θ_{ori} with more details in the supplement.

Similarly, the equation of orientation θ_{ori} is derived as:

$$\frac{\partial \theta_{ori}}{\partial t} = -M_{ori} J (1 - p(\zeta)) \nabla_s \cdot \left(p(\zeta) \frac{\nabla_s \theta_{ori}}{|\nabla_s \theta_{ori}|} \right), \quad (7)$$

in which M_{ori} and J are constants, and M_{ori} describes the orientation mobility. Considering the environment temperature T_{env} using Newton's law of cooling, the evolution of temperature T is determined based on the temperature equation proposed by Kobayashi [1993]:

$$\frac{\partial T}{\partial t} = a^2 \nabla_s^2 T + K \frac{\partial \zeta}{\partial t} + \frac{1}{\tau} (T_{env} - T), \quad (8)$$

where a and K are constants that affect heat diffusion and the latent heat released by solidification, while τ is a time constant. As shown in Fig. 4, we highlight the Marangoni effect (i.e., significant surface flow) induced by the temperature changes from released latent heat during solidification. More implementation details are included in the supplement.

4 PHASE MAP

In this section, we first discuss the issue of naively integrating MELP with the previously mentioned orientation-based dendritic crystal simulation method. Then, we propose the Phase Map method to resolve the issue. The pipeline of the Phase Map method is shown in Fig. 2.

4.1 Issue in the Naive Integration

A naive integration would be to calculate the temperature and phase equations on either the $\mathcal{E}$ particles or the $\mathcal{L}$ particles. Since the $\mathcal{E}$ particles are too sparse to reproduce the detailed dendritic shapes, it will seem better to use $\mathcal{L}$ particles to update the temperature T and phase ζ . However, as depicted on the top of Fig. 18, the dendrites fail to maintain the detailed branches shown by Ren et al. [2018]. There are two main reasons for this result: 1) the uneven distribution of particles (manifested by the irregular particle gaps



Fig. 5. Dendritic crystals grow, move, and swirl with surface flow on a floating bubble. **Top:** the bubble case. **Bottom:** the ellipsoid case.

in Fig. 18) caused by the arbitrary motion of the $\mathcal{L}$ particles hinders the accurate calculation of orientation and phase changes due to erroneous gradients; 2) the momentum motion of the $\mathcal{L}$ particles does not consider particle rotations, and the relative motion at the solid-liquid interface alters the direction of freeze fronts. The local orientation field θ_{ori} cannot capture the actual molecular orientation, leading to distortions of detailed patterns resembling the trail-like branches shown in the top row of Fig. 18.

4.2 Concept of Phase Map

Inspired by MELP, we further exploit their core idea of separating particles for different calculation tasks by employing a third set of particles $\mathcal{M}$. We term the new scheme as Phase Map and the $\mathcal{M}$ particles serve to avoid the distortion mentioned above and track the dendrite growth. In addition, we define a set of the dendritic crystals $\mathcal{C}$, where each crystal $C \in \mathcal{C}$ comprises crystallized $\mathcal{L}$ particles. We associate with each C a block Ω which is composed of a set of uniformly distributed $\mathcal{M}$ particles (as illustrated in Fig. 2) and are independent of the bubble or film. For each set of $\mathcal{M}$ particles, physical quantities of the corresponding crystal and its neighborhood particles (including those of nearby fluid and other crystals) on $\mathcal{L}$ and $\mathcal{M}$ can communicate using $\mathcal{L}2\mathcal{M}$ and $\mathcal{M}2\mathcal{L}$ Transfer. During initialization, we randomly sample crystallization centers on $\mathcal{L}$ as the crystal set $\mathcal{C}$ at time 0, with the number of the crystallization centers matching the count of the blocks on $\mathcal{M}$. Based on the fact that crystals are all grown from the crystallization centers, we treat these crystallization centers as the anchor points, and link them to the centers of the corresponding blocks. During simulation, the $\mathcal{M}$ particles remain uniformly distributed and fixed in position, and are linked to the deformed surface represented by $\mathcal{E}$ particles through the mapped positions and rotations stored at the crystallization

centers. This treatment ensures accurate SPH discretization and stable global orientation on $\mathcal{M}$.

For each $C \in \mathcal{C}$, the corresponding block Ω on $\mathcal{M}$ creates a "sandbox", and the dendritic crystal can be simulated to evolve in the sandbox. The result of the evolution serves as a template that guides the update of ζ , T , and θ_{ori} on $\mathcal{L}$. We denote the $\mathcal{L}$ particles from fluid and other crystals surrounding C as $\mathcal{C}_e$. Specifically, we first need a Phase Mapping step to align C and $\mathcal{C}_e$ to Ω , and place the dendritic crystals into the sandbox. Then, we introduce the effect of $\mathcal{C}_e$ to the sandbox via the $\mathcal{L}2\mathcal{M}$ step. Subsequently, the dendrite crystals grow on the sandbox (Freezing Computation step). Finally, the result of the growth is taken out of the sandbox through the $\mathcal{M}2\mathcal{L}$ step.

The freezing equations (Eq. (6), (7) and (8)) are solved on $\mathcal{M}$, with ζ , T , θ_{ori} transferred between the $\mathcal{L}$ and $\mathcal{M}$ particles. As shown in Fig. 2, our freezing algorithm consists of four main steps: (a1) Phase Mapping, (a2) $\mathcal{L}2\mathcal{M}$ Transfer, (a3) Freezing Computation, (a4) $\mathcal{M}2\mathcal{L}$ Transfer.

4.3 Phase Mapping

For each $C \in \mathcal{C}$, we compute the 2D rotation angle β_C as:

$$\beta_C = \arctan2 \left(\sum_{L \in C} \sin(\beta_L), \sum_{L \in C} \cos(\beta_L) \right), \quad (9)$$

where L is one of the Lagrangian particles that constitute C , and β_L is L 's 2D rotation angle of the state represented by $\mathcal{M}$ to the current state.

Then, we compute all possible mapped positions for each $L \in \mathcal{L}$. If L is close enough to $C \in \mathcal{C}$, we compute the mapped position $\mathbf{x}_L^C$ as:

$$\mathbf{x}_L^C = R(\beta_C) \top_C (\mathbf{x}_L - \mathbf{x}_C) + \mathbf{x}_C^C, \quad (10)$$

where $R(\beta_C)$ stands for the rotation matrix of β_C , and $\mathbf{x}_C^C$ is the mapped position of the crystallization center of C .

4.4 $\mathcal{L}2\mathcal{M}$ Transfer

We divide $\mathcal{M}$ into two parts: the template part $\mathcal{M}_t$ and the environment part $\mathcal{M}_e$. To eliminate part of the environmental influence that breaks the shape of dendritic crystals, $\mathcal{M}_t$ does not collect quantities from nearby Lagrangian particles mapped from $\mathcal{L}$. At the same time, $\mathcal{M}_e$ receives quantities from the environment to ensure that the template freezes properly when multiple freeze fronts meet. In practice, we use the spatially weighted average to ensure that the physical quantity q (ζ or T) is normalized after the transfer from $\mathcal{L}$ to $\mathcal{M}$, which is defined as:

$$\mathcal{T}_{\mathcal{M} \leftarrow \mathcal{L}}(q) = \frac{\sum_{L \in \mathcal{N}^{\mathcal{L}}(\mathcal{M})} W(M, L) \cdot q_L}{\sum_{L \in \mathcal{N}^{\mathcal{L}}(\mathcal{M})} W(M, L)}, \quad (11)$$

in which $\mathcal{N}^{\mathcal{L}}(\mathcal{M})$ (all $\mathcal{L}$ neighbors in the neighborhood of $M \in \mathcal{M}_e$) are obtained using the mapped positions $\mathbf{X}$. Thus $\zeta_M = \mathcal{T}_{\mathcal{M} \leftarrow \mathcal{L}}(\zeta)$, $T_M = \mathcal{T}_{\mathcal{M} \leftarrow \mathcal{L}}(T)$, and $(\theta_{ori})_M = (\theta_{ori})_{L_n} + \beta_C$, where L_n is the Lagrangian particle nearest to M , and C corresponds to the crystal template to which M belongs.

4.5 Freezing Computation

For each $M \in \mathcal{M}_t$, we update ζ , θ_{ori} , and T according to Eq. (6), (7) and (8), respectively. It is worth noting that the computations above are using $\mathcal{N}^{\mathcal{M}}(\mathcal{M})$ within the whole $\mathcal{M}$ context to account for the influence of the environment.

4.6 $\mathcal{M}2\mathcal{L}$ Transfer

Similar to the $\mathcal{L}2\mathcal{M}$ step, the transfer for each $L \in \mathcal{L}$ is performed as: $\zeta_L = \mathcal{T}_{\mathcal{L} \leftarrow \mathcal{M}}(\zeta)$, $T_L = \mathcal{T}_{\mathcal{L} \leftarrow \mathcal{M}}(T)$, and $(\theta_{ori})_L = (\theta_{ori})_{M_n} - \beta_C$, where C represents the crystal to which L belongs, $\mathcal{N}^{\mathcal{M}}(\mathcal{L})$ is obtained using the mapped positions $\mathbf{x}_L^C$, and M_n is the particle nearest to $\mathbf{x}_L^C$ on $\mathcal{M}$. For those uncrystallized $\mathcal{L}$ particles located within overlapping regions influenced by multiple crystals, they collect the quantities from the block on $\mathcal{M}$ corresponding to the nearest crystal.

4.7 Artistic Crystal Pattern Control

Building upon the approach outlined by Ren et al. [2018], our method inherently allows artists to edit the orientation field and thus control dendrite shapes. Furthermore, by utilizing Phase Map, we can artistically control the ice crystal shape not limited to dendrite shapes. For example, we can use Signed Distance Functions (SDF) as designated ice crystal shapes. In practice, we remove the $\mathcal{M}2\mathcal{L}$ step and the solving of Eq. (6). Instead, we employ the SDF field as the phase field by inverting the negative SDF values and normalizing them to $[0, 1]$. Additionally, the positive SDF values are deprecated and the phase values are not allowed to decrease in order to prevent visual flicker. The $\partial\zeta/\partial t$, which is used in Eq. (7) and (8), is calculated based on the change in the phase field value.

5 BOUNDARY HANDLING FOR COMPLEX PHASE COUPLING

MELP uses several layers of boundary particles $\mathcal{B}$ to resolve the particle insufficiency near the boundary and cannot handle the fluid-solid coupling physically, which is an inherent limitation of the framework. In our scenarios, the freezing process introduces complex coupling of the fluid, dendritic crystals, and the boundary, which requires accurate boundary handling.

Inspired by the MLS pressure boundaries method for SPH [Band et al. 2018], we derive the MLS concentration boundaries for MELP based on the fact that the surfactant concentration Γ plays a pivotal role in the dynamics computation.

To estimate the surfactant concentration Γ on $\mathcal{B}$, we define the fitting function $\tilde{\Gamma}_B(\mathbf{x}) = \mathbf{b}(\mathbf{x}) \cdot \mathbf{c}_B$ for each $B \in \mathcal{B}$, where $\mathbf{b}(\mathbf{x}) = (1, x, y)^T$ is the linear basis and $\mathbf{c}_B = (\alpha_B, \beta_B, \gamma_B)^T$ is the vector of all basis coefficients that fit B 's hyperplane. Thus, we formulate our problem as:

$$\text{minimize} \sum_{E \in \mathcal{N}^{\mathcal{E}}(B)} a_E \left(\tilde{\Gamma}_B(\mathbf{x}_E) - \Gamma_E \right)^2 \bar{W}_{BE}, \quad (12)$$

in which $\mathcal{N}^{\mathcal{E}}$ are all $\mathcal{E}$ neighbors in the neighborhood of B .

To solve Eq. (12), we consider our 3×3 system on the surface as:

$$\left(\sum_{E \in \mathcal{N}^{\mathcal{E}}(B)} a_E \begin{bmatrix} 1 & 0 \\ 0 & G_E \end{bmatrix} \bar{W}_{BE} \right) \begin{bmatrix} \alpha_B \\ \beta_B \\ \gamma_B \end{bmatrix} = \sum_{E \in \mathcal{N}^{\mathcal{E}}(B)} a_E \begin{bmatrix} 1 \\ \bar{u}_E \\ \bar{v}_E \end{bmatrix} \Gamma_E \bar{W}_{BE}. \quad (13)$$

In the above equation, $\bar{u}_E$ and $\bar{v}_E$ are the components of $\tilde{\mathbf{x}}_E^R = (\bar{u}_E, \bar{v}_E)^T = \mathbf{x}_E^R - \mathbf{d}_B$, where $\mathbf{x}_E^R$ is E 's coordinate on the tangent plane of B 's local frame R , $G_E = (\bar{u}_E, \bar{v}_E)^T (\bar{u}_E, \bar{v}_E)$, and $\mathbf{d}_B$ is given as:

$$\mathbf{d}_B = \frac{\sum_{E \in \mathcal{N}^{\mathcal{E}}(B)} a_E \mathbf{x}_E^R \bar{W}_{BE}}{\sum_{E \in \mathcal{N}^{\mathcal{E}}(B)} a_E \bar{W}_{BE}}. \quad (14)$$

The coefficients are computed as:

$$\alpha_B = \frac{\sum_{E \in \mathcal{N}^{\mathcal{E}}(B)} a_E \Gamma_E \bar{W}_{BE}}{\sum_{E \in \mathcal{N}^{\mathcal{E}}(B)} a_E \bar{W}_{BE}}, \quad (15)$$

$$\begin{bmatrix} \beta_B \\ \gamma_B \end{bmatrix} = \left(\sum_{E \in \mathcal{N}^{\mathcal{E}}(B)} a_E G_E \bar{W}_{BE} \right)^{-1} \sum_{E \in \mathcal{N}^{\mathcal{E}}(B)} a_E \tilde{\mathbf{x}}_E^R \Gamma_E \bar{W}_{BE}. \quad (16)$$

Finally, we obtain $\Gamma_B = (1, \bar{u}_B, \bar{v}_B)^T \cdot (\alpha_B, \beta_B, \gamma_B)^T$ and Γ_B can be used as pseudo surfactant concentration when solving Γ on $\mathcal{E}$ particles near boundaries according to Eq. (4).

We also approximate the pseudo-pressure C mentioned in $\mathcal{E}$ Advance step to ensure that $\mathcal{E}$ particles are uniformly distributed. Similarly, the basis coefficients $\omega_B, \theta_B, \phi_B$ are given as:

$$\omega_B = \frac{\sum_{E \in \mathcal{N}^{\mathcal{E}}(B)} a_E C_E \bar{W}_{BE}}{\sum_{E \in \mathcal{N}^{\mathcal{E}}(B)} a_E \bar{W}_{BE}}, \quad (17)$$

$$\begin{bmatrix} \theta_B \\ \phi_B \end{bmatrix} = \left(\sum_{E \in \mathcal{N}^{\mathcal{E}}(B)} a_E G_E \bar{W}_{BE} \right)^{-1} \sum_{E \in \mathcal{N}^{\mathcal{E}}(B)} a_E \tilde{\mathbf{x}}_E^R C_E \bar{W}_{BE}, \quad (18)$$

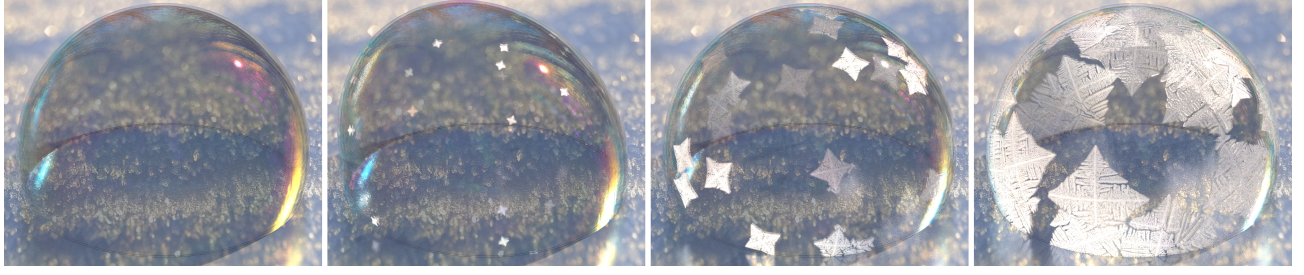


Fig. 6. A dome-like bubble freezes and sways with the horizontal sweeping wind.

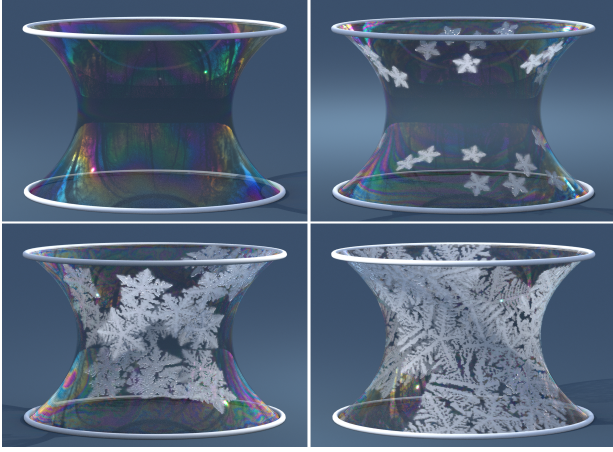


Fig. 7. Dendritic crystals travel around the catenoid during the freezing process.

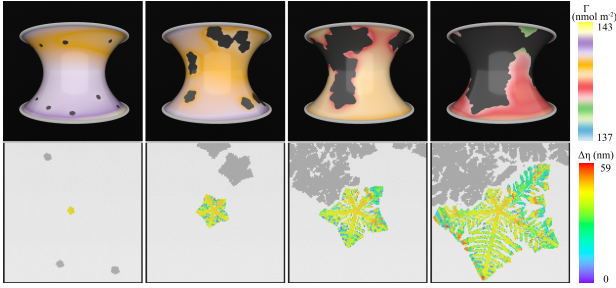


Fig. 8. **Top:** Visualization of the surfactant concentration, in which crystals are black and fluids are other colors. As illustrated in the figures, the surfactant concentration near the crystals decreases. **Bottom:** Visualization of the changes in the local thickness of the crystal, only one block in $\mathcal{M}$ is shown for clarity. Fluids are visualized in white, the surrounding crystals are shown in gray, and the crystal corresponding to the block is displayed in other colors.

$$\text{and } C_B = (1, \bar{u}_B, \bar{v}_B)^T \cdot (\omega_B, \theta_B, \phi_B)^T.$$

6 IMPLEMENTATION

We adopt a Lagrangian perspective to handle the advection term, updating the positions of $\mathcal{L}$ particles. The evolution of phase field

and temperature is resolved on $\mathcal{M}$ particles, generating more detailed dendritic crystal shapes. We address the projection term on sparser, uniformly distributed $\mathcal{E}$ particles using Implicit Incompressible SPH (IISPH) [Ihmsen et al. 2013]. Material and momentum transfer between $\mathcal{E}$ and $\mathcal{L}$ is facilitated through Affine Particle-In-Cell (APIC) [Jiang et al. 2015]. For more details on discretization, we refer readers to the supplement.

Overview. In each time step, we begin by updating the phase, temperature, and orientation fields via the phase map. Subsequently, we transfer the necessary quantities from $\mathcal{L}$ to $\mathcal{E}$ for the thin film computation. After solving the projection term on $\mathcal{E}$ and interpolating velocity from $\mathcal{E}$ to $\mathcal{L}$, we address the fluid-solid coupling by calculating the linear and angular resultant velocity of each ice dendrite on $\mathcal{L}$. Finally, we advect $\mathcal{E}$ and then $\mathcal{L}$. The following comprehensive overview includes all the necessary components in our framework.

- (a) **Phase Map Computation:** Update phase ζ , temperature T , and orientation θ_{ori} on $\mathcal{M}$ (Section 4.3, 4.4, 4.5 and 4.6).
- (b) **$\mathcal{L}2\mathcal{E}$ Transfer:** Transfer necessary quantities from $\mathcal{L}$ to $\mathcal{E}$. For a generic quantity q , such as mass m and volume V , we utilize a conservative partition of unity transfer as in the MELP method. Moreover, we employ the spatially weighted average $\mathcal{T}_{\mathcal{E} \leftarrow \mathcal{L}}(T)$ to transfer T .
- (c) **Thin Film Computation:** Update thickness η for $\mathcal{L}$ and $\mathcal{E}$; update control area a , mean curvature H and metric tensor g for $\mathcal{E}$. Solve Eq. (3) on $\mathcal{E}$ and update velocity $\mathbf{u}$.
- (d) **$\mathcal{E}2\mathcal{L}$ Transfer:** $\mathcal{L}$ collect $\mathbf{u}$ from neighboring $\mathcal{E}$.
- (e) **Fluid-solid Coupling:** Calculate the resultant mass, momentum, and centroid of each ice dendrite. Then, determine the resultant linear and angular velocity of the frozen $\mathcal{L}$ particles.
- (f) **Advance:** Advect $\mathcal{E}$ particles and adjust their positions to maintain uniform distribution on the updated S_B . Advect $\mathcal{L}$ particles and then project them onto S_B .

The number/area of $\mathcal{M}$. To ensure that each ice dendrite grows independently, each block on $\mathcal{M}$ is isolated from each other. We set $\mathcal{M}$ with the same fineness as $\mathcal{L}$ to ensure uniform resolution and SPH kernel size. As mentioned in Section 4.2, the effect of the surroundings for each $C \in \mathcal{C}$ is introduced to blocks on $\mathcal{M}$. It is worth noting that multiple dendritic crystals may have overlapping surroundings, so the blocks on $\mathcal{M}$ need to be sufficiently redundant. In our practice, the total number/area of the $\mathcal{M}$ particles needs to be initialized to 2.5 to 3.5 times that of the $\mathcal{L}$ particles.

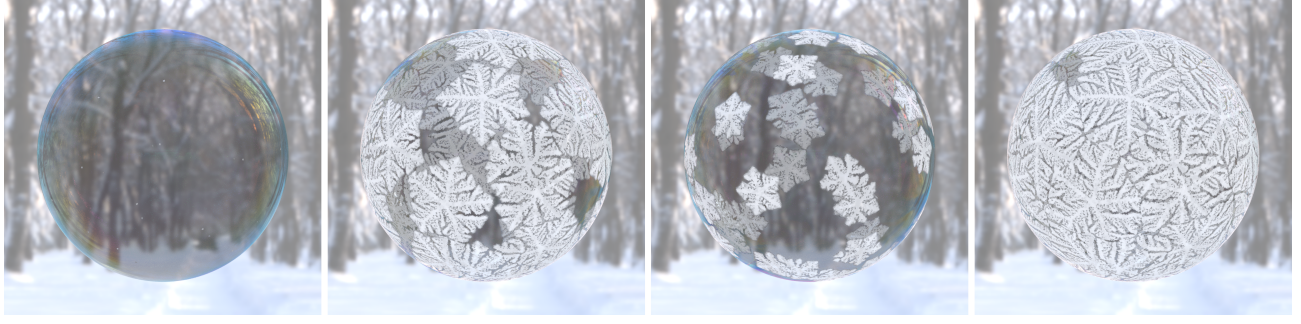


Fig. 9. Symmetry breaking effects for 5-fold patterns using $M_{ori} = 2500$.

Time Step. Given that the freezing computation necessitates a smaller time step compared to the original MELP computations, we integrate several freezing substeps (typically 10) within a single MELP step, ensuring an equal total time for both dynamics and freezing processes.

Particle Division for $\mathcal{L}2M$ Transfer. As mentioned in Section 4.4, we divide the M particles into the template part M_t and the environment part M_e . We add a particle of M to M_t if it is covered by the range of influence of the solid particles, otherwise we add it to M_e . In addition, if multiple dendritic crystals are close enough, their range of influence may overlap, so in this case, we add only one layer of M around the solid particles to M_t , while others are added to M_e .

Phase Change and Coupling. For $\mathcal{E}$ particles, we do not explicitly distinguish between solid and liquid phases, nor do we consider collisions between ice dendrites and the boundary. As for $\mathcal{L}$ particles, we differentiate ice dendrites (the solid phase) from the liquid phase based on the phase value ζ , utilizing a threshold φ_{den} . The φ_{den} lies in the range $(0, 1)$, and we typically set it to 0.1, which allows the liquid particles near the ice dendrites to move together, thereby preventing significant deviations in the shape of ice dendrite growth. Once the phase value exceeds the threshold in the Freezing step (i.e., $\zeta \geq \varphi_{den}$), liquid particles transform into solid particles. For solid handling, we assume that each $C \in \mathcal{C}$ is a rigid body. In addition, we merge adjacent dendritic crystals if their relative velocity is small enough in magnitude, thereby preventing excessive severe motion of individual dendritic crystals. Subsequently, the total mass, total momentum, and centroid of each C are calculated. Using the centroid, we obtain the total angular momentum and the moment of inertia. Finally, we compute the linear and angular velocity of rigid bodies, along with the final velocity $\mathbf{u}_L^{**}$ of the Lagrangian particle L . More implementation details can be found in the supplement. In the $\mathcal{L}$ advance step, we update position $\mathbf{x}_L$ using $\mathbf{u}_L^{**}$.

7 EXPERIMENTS

7.1 Examples

In this section, we show some examples of freezing on thin films under different conditions. We implement the method in parallel on an NVIDIA GeForce RTX 4080 GPU, with an Intel Core i9-13900K CPU. The computational cost of the examples is listed in the table 2,

and the parameters used in our examples are provided in the supplement. For all examples, the ambient temperature is -20°C , and J is 10^{-5} . The frame size is 0.004 s and each frame may contain several MELP steps mentioned in Section 6 to match the frame size.

Floating Bubble. In this example, we place a bubble with a radius of 0.1m in the air. In the top row of Fig. 5, the bubble is initialized with η disturbed by noise and uniform θ_{ori} . We adopt a similar hear-driven buoyancy to the flow motion as MELP [Deng et al. 2022] in the examples. We use $K = 3.7$ to create detailed branches. We also initialized an ellipsoidal bubble with a semi-minor axis of 0.1 m and a semi-major axis of 0.14 m, and applied a random displacement to the bubble, as illustrated in the bottom row of Fig. 5. Additionally, we set the dendritic crystals to 8-fold patterns, distinguishing it from the previous case. The dendritic crystals show significant motion and swirling with the flow. At the same time, thanks to our Phase Map scheme, the dendritic crystals preserve the patterns and details similar to results shown by Ren et al. [2018].

Swaying Dome-like Bubble. Fig. 6 shows that a dome-like bubble freezes of radius 0.26 m in the snow. Thickness is perturbed using Perlin noise in the initialization. A horizontal sweeping wind field similar to that used in MELP [Deng et al. 2022] is applied to the scene for the first 250 frames only, making the bubble sway with the wind. K is set to 3.7 and dendritic crystals are set to 4-fold patterns in this example. The example demonstrates that our method is capable of handling the freezing process of bubbles with complex surface flow and evolving topology. The particle resolution is raised compared to other experiments (as is shown in the computational cost Table provided in the supplement), it can be observed that as the resolution goes up fine details such as minor branch directions are recovered better, at the cost of more computational time.

Catenoid. As shown in Fig 7, a catenoid is initialized between two parallel circular rims. The neck radius and height of the catenoid are 0.07 m and 0.18 m, respectively. The initial thickness is also perturbed using Perlin noise. The buoyancy force is applied horizontally in this example. This example demonstrates the applicability of our method for complex situations. The visualizations of the surfactant concentration and the local thickness of the crystals are shown in the top and bottom rows of Fig. 8, respectively. In the figure, the surfactant concentration near the crystals decreases, reflecting the influence of the crystals on the fluids. The changes in the local



Fig. 10. Dendritic crystals grow into spiral patterns according to the fixed orientation field (visualized as the grayscale image on the far right).

thickness of the crystal are roughly consistent with the phase field distribution. It is to be noted that due to the perturbations in the initial fluid thickness, the changes exhibit some unevenness and irregularity.

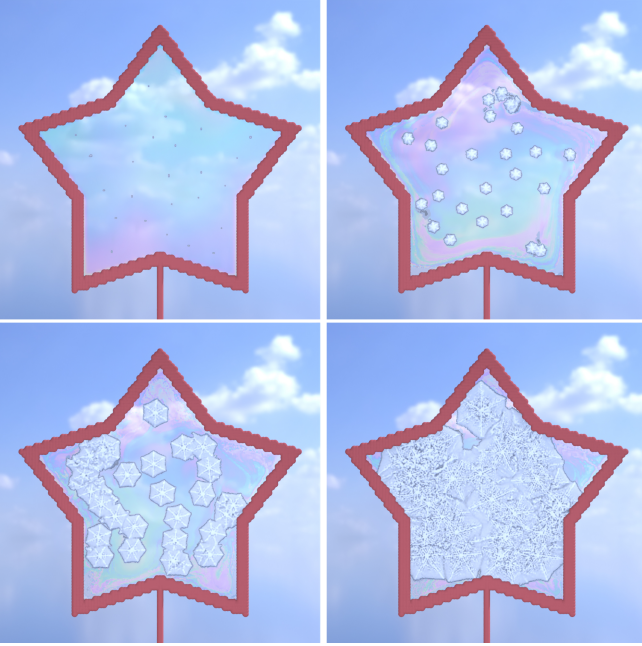


Fig. 11. Different frames of a freezing thin film inside the 5-point star-shaped boundary, with a rotational wind field.

Symmetry Breaking. As depicted in Fig. 9, the example shows the symmetry breaking effects during the growth process of dendritic crystals. The settings are inherited from the bubble case in the Floating Bubble example, while the patterns are changed to 5-fold, and M_{ori} is 2500 for obvious symmetry breaking. Imperfect dendritic crystal shapes enable our method to create more realistic freezing effects.

Spiral Dendritic Crystals. Fig. 10 shows the freezing process guided by fixed orientation. In this scenario, we initialize a dome-like bubble with a radius of 0.14 m, and perturb its thickness with Perlin

Table 2. Computational cost of our examples.

Figure	Example	# $\mathcal{L}$	# $\mathcal{M}$	# $\mathcal{E}$	Time/frame (s)
5	Floating Bubble (Sphere)	320k	1M	19k	1.77
5	Floating Bubble (Ellipsoid)	320k	1M	19k	1.72
6	Swaying Dome-like Bubble	1.5M	2.6M	87k	37.9
7	Catenoid	164k	493k	9k	1.36
9	Symmetry Breaking	320k	1M	19k	2.09
10	Spiral Dendritic Crystals	416k	1.6M	25k	2.33
11	5-point Star-shaped Thin Film	73k	1M	5k	1.30
12	Hearts on a Bubble	320k	914k	19k	0.89
13	Flowers on a Thin Film	80k	640k	5k	0.49

noise. As displayed in the far right of Fig. 10, we fix the orientation field through a grayscale map, in which color indicates direction and gradients radially. The dendritic crystals grow into spiral patterns during movement, creating a beautiful artistic effect.

5-point Star-shaped Thin Film. As shown in Fig. 11, we initialize a 5-point star-shaped thin film with the radius of the circumscribed circle set to 0.2 m. A rotational wind field is applied during the first 300 frames to align with the video captured in the real world, and the comparison is shown in the supplemental video. We use $K = 3.0$ to inhibit the growth of branches. Utilizing the MLS concentration boundaries, our framework shows great robustness when handling complex boundary and external force conditions.

Hearts on a Bubble. As depicted in Fig. 12, the configuration is almost the same as the sphere case in the Floating Bubble example except for the way the ice dendrites grow. We compute and normalize the SDFs of hearts over time, and update ζ on $\mathcal{M}$ particles according to the SDFs. Our approach can naturally integrate artistic controlled patterns into the freezing process.

Flowers on a Thin Film. As shown in Fig. 13, a 2D bubble film is placed in a circular rim and initialized to be thick at the top ($\eta \approx 300$ nm) and thin at the bottom ($\eta \approx 550$ nm). The user-controlled SDFs are designed to guide the ice dendrites to grow petals. The results show the interaction of the artistically designed dendritic crystals and the pattern of Rayleigh-Taylor instability.

7.2 Comparisons

Phase Map. We first validate our proposed Phase Map scheme. In order to demonstrate the indispensability of our Phase Map method,

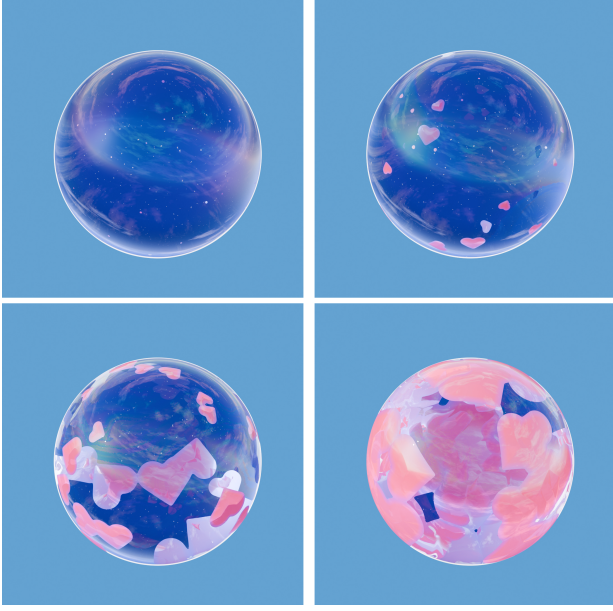


Fig. 12. Heart-shaped dendritic crystals on a floating bubble, rendered in a non-realistic style.

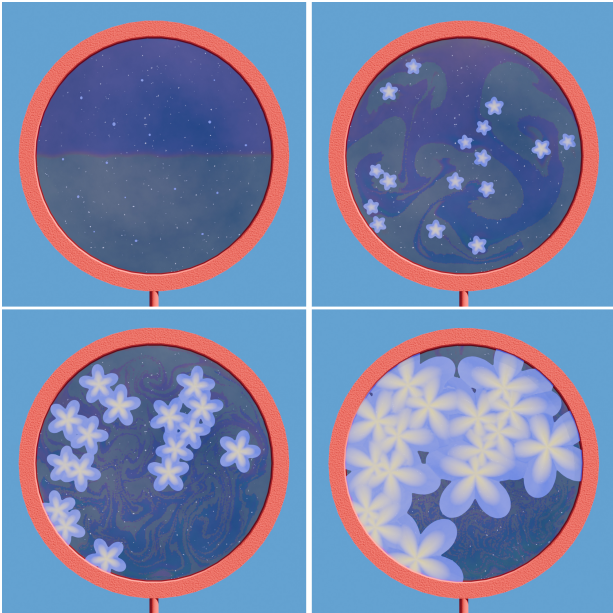


Fig. 13. Flower-shaped dendritic crystals and Rayleigh-Taylor instability on a thin film inside a circular rim.

we conduct a comparison validation between the dendrite shape on moving particles with and without using Phase Map. On the top of Fig. 18, without using Phase Map scheme, the dendritic crystal shapes are distorted by the arbitrary particle distribution, while the result of using Phase Map on the bottom shows the correct shape.

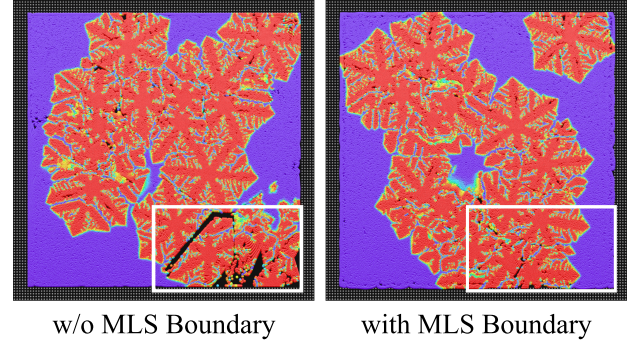


Fig. 14. Comparison of boundary handling strategies with (right) and without (left) using MLS concentration boundaries. The boundary is black and the fluid is blue.

MLS Concentration Boundaries. To make our simulation process of two-phase coupling stable, we apply the MLS concentration boundaries. We validate this by simulating a thin film freezing process with and without using MLS boundaries. As shown in Fig. 14, the use of the MLS boundary method effectively avoids the artifacts near the boundary.

Isothermal Conditions. In isothermally supercooled bubbles, the latent heat released by the freezing process engenders a robust Marangoni flow that entrains ice crystals, resulting in the aforementioned snow globe effect. Our result under isothermal conditions compared to the real world can be seen in Fig. 15 for temperature.

Room-temperature Conditions. Utilizing a cold stage in a warm ambient environment leads to a bottom-up freeze front that eventually halts due to poor conduction along the bubble. To simulate the process, we place an overcooled bubble on a cold substrate (40°C), with an ambient temperature of 20°C . Four layers of the bottom particles of the $3/4$ bubble are initialized with $\zeta = 1$. The comparison to the real world can be seen in Fig. 16 for temperature and Fig. 17 for the rendered result.

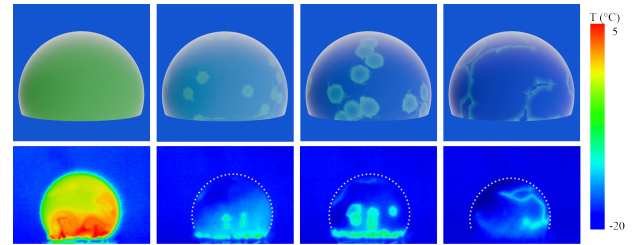


Fig. 15. Comparison of bubble temperatures under isothermal conditions. **Top:** our results which ignore the air temperature inside the bubble. **Bottom:** thermographic images shown by Ahmadi et al. [2019].

8 CONCLUSIONS AND LIMITATIONS

We have introduced a novel Phase Map method for the controllable simulation of complex freezing dynamics on thin films. Based upon the derivation of temperature-related governing equations, we have

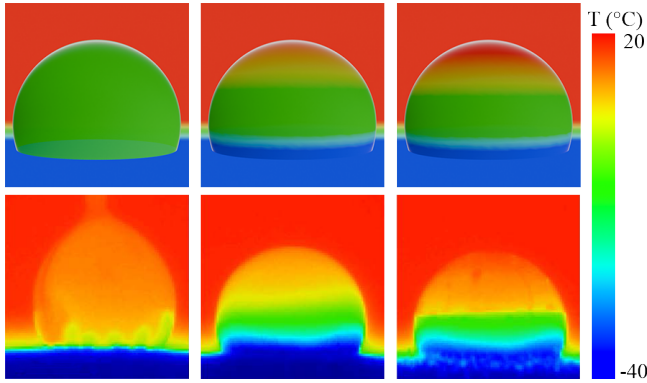


Fig. 16. Comparison of bubble temperatures at room temperature. **Top:** our results which ignore the air temperature inside the bubble. **Bottom:** thermographic images shown by Ahmad et al. [2019].



Fig. 17. Render results of our method (top) at room temperature compared to reality [Ahmad et al. 2019] (bottom).

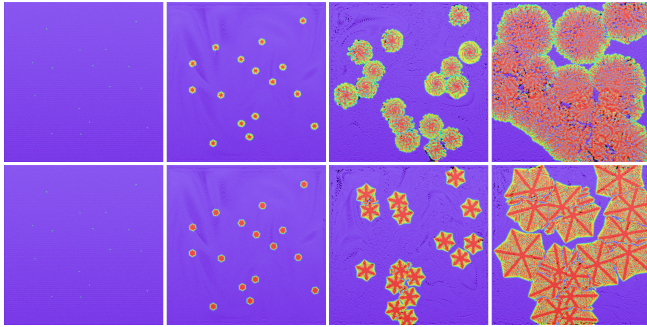


Fig. 18. We validate our proposed Phase Map scheme by conducting a comparison between the dendrite shape on moving particles with and without using Phase Map. **Top:** distorted dendrite shape without using Phase Map. **Bottom:** detailed dendrite shape using Phase Map.

built the freezing dynamics simulation framework on top of MELP to capture the thermodynamic process as thin films freeze. We have utilized the 'sandbox' concept, to preserve structured freezing patterns over unstructured moving particle representation and have enabled artistic control over diverse visual simulations. Meanwhile,

the derived MLS concentration boundary has significantly enhanced our framework's capability to deal with complex boundaries.

The main limitations of our work lie in 1) The inter-phase coupling in our method can be improved. The ice detachment from the contact line caused by Marangoni flow is not able to be reproduced. We can explore more accurate fluid-solid and solid-solid coupling in future work to obtain more realistic results. 2) Although the MELP method can simulate multi-bubbles, we haven't validated our method on Multi-MELP. We can explore the multi-bubble freezing process involving ice dendrites transferred between connected bubbles, creating appealing phenomena that barely be seen in the real world. 3) The current approach has relatively low memory efficiency, so we would like to investigate more memory-friendly implementations in the future.

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